The Kosterlitz-Thouless phase in a hierarchical model

1990 J. Phys. A: Math. Gen. 23 1207

(http://iopscience.iop.org/0305-4470/23/7/025)

J. Phys. A: Math. Gen. 23 (1990) 1207-1215. Printed in the UK

The Kosterlitz–Thouless phase in a hierarchical model

J Dimock

Department of Mathematics, SUNY at Buffalo, Buffalo, NY 14214, USA

Received 4 July 1989

Abstract. We use renormalisation group methods to study the long-distance behaviour of hierarchical $(\cos\varphi)_2$ lattice field theories in the Kosterlitz-Thouless phase.

1. Introduction

We consider $(\cos\varphi)_2$ (sine–Gordon) field theories on a lattice. For the lattice we take the two-dimensional toroidal lattice $\Lambda=\Lambda_N=(\mathbb{Z}/L^N\mathbb{Z})^2$ with fixed integer $L\geqslant 2$. The free field theory on Λ is given by the Gaussian measure μ on $\mathbb{R}^\Lambda$ with covariance $\beta(-\Delta)^{-1}$ where $\beta>0$ and Δ is the lattice Laplacian. For $f\in\mathbb{R}^\Lambda$ we have

$$\int e^{i(\varphi,f)}\,d\mu(\varphi)=\exp[-\tfrac{1}{2}\beta(f,(-\Delta)^{-1}f)]. \tag{1.1}$$

Actually $(f,(-\Delta)^{-1}f)$ is well defined only if $\sum_x f(x)=0$; otherwise we interpret it as ∞, and the right-hand side of (1.1) as zero. (One can think of using $(-\Delta+\xi)^{-1}$ as the covariance and taking the limit $\xi\to 0$.)

For the full theory we consider the measure

$$e^{-V(\varphi)}\,d\mu(\varphi) \tag{1.2}$$

where

$$V(\varphi)=2z\sum_{x\in\Lambda}\cos\varphi(x) \tag{1.3}$$

with $z\geqslant 0$.

As is well known, this model can be thought of as describing a classical lattice gas of charged particles with Coulomb interaction $(-\Delta)^{-1}$ and an overall neutrality condition. For example, by expanding the exponential one has

$$\int e^{-V(\varphi)}\,d\mu(\varphi)=\sum_{n=0}^{\infty}\frac{z^n}{n!}\left[\sum_{\substack{x_1,\ldots,x_n\\ q_1,\ldots,q_n}}\exp\left(-\frac{\beta}{2}\sum_{i,j}q_iq_j(-\Delta)^{-1}(x_i,x_j)\right)\right] \tag{1.4}$$

where the sum is over $x_i\in\Lambda$, $q_i=\pm 1$ with the restriction $\sum q_i=0$. This is the partition function for the gas in the grand canonical ensemble with activity z and temperature β^{-1}.

A general discussion of results for the model can be found in [1].

For small z there are two distinct phases (and possibly many more [2]) depending on β. For small β (high temperature) there is a plasma phase. In this phase correlations show an exponential decay corresponding to a Debye screening phenomena. This has been rigorously studied by Brydges and Federbush [3, 4].

0305-4470/90/071207+09$03.50

On the other hand, for large β (low temperature) there is a phase in which the charged particles form dipoles. In this phase correlations show a power-law decay and there is no Debye screening. The phase was discovered by Kosterlitz and Thouless [5]. A proof that this Kosterlitz-Thouless phase exists was given by Fröhlich and Spencer [6].

Our interest is in extending the results of Fröhlich and Spencer by giving a more systematic renormalisation group analysis and obtaining more detailed information on the decay of correlations. Formal renormalisation group treatments have appeared in the literature, e.g. [7]. A rigorous version could follow the lines laid down by Gawedzki and Kupiainen who consider (among other things) a dipole gas [8-10]. However, the extension of their methods to this problem is not at all straightforward. So here we are content to start with a renormalisation group analysis for a simpler hierarchical version of the model. The study of hierarchical models by renormalisation group methods has also been developed by Gawedzki and Kupiainen [9-12], and has proved to be a fruitful line of attack.

For the hierarchical model there is a natural sequence of effective measures for increasing length scales. Our basic result is that they become increasingly Gaussian, i.e. we have infrared asymptotic freedom (theorem 1). This determines the long-distance behaviour of the theory. As an illustration, we consider a certain fractional charge correlation function $\mathcal{G}(x, y)$. For this function we obtain the infinite-volume limit $N \to \infty$ and then find the exact asymptotic behaviour as $|x-y| \to \infty$ (theorem 2).

Our methods generally are those of Gawedzki and Kupiainen. The main difference is that we systematically use Fourier expansions rather than Taylor expansions to analyse the effective interactions.

2. The model and the renormalisation group

The hierarchical approximation consists in replacing the covariance $\beta(-\Delta)^{-1}$ of the Gaussian measure by βC where

$$C(x, y) = N - K(x, y) \qquad K(x, y) = \inf\{k \in \mathbb{Z}, k \geq 0: [L^{-k}x] = [L^{-k}y]\} \tag{2.1}$$

with [ ] denoting the integral part.

This covariance preserves the essential features of the problem. With an l^∞ metric on Λ we have $K(x, y) \geq \log_L|x-y|$, and for most x, y, $K(x, y)$ is close to $\log_L|x-y|$. Thus $-K(x, y)$ has the same type of long-distance behaviour as the inverse Laplacian. Note that for $\sum f = 0$ we have $(f, Cf) = (f, (-K)f)$ so it is $-K$ that is relevant. On the other hand, if $\sum f \neq 0$ then (f, Cf) is infinite (in the limit $N \to \infty$), just as for the original problem.

The hierarchical model can be realised by writing

$$\varphi(x) = \sum_{k=0}^{N-1} Z^k([L^{-k-1}x]) \tag{2.2}$$

where $\{Z^k(u)\}$ for $0 \leq k \leq N-1$ and $u \in \mathbb{R}^{\Lambda_{N-k-1}}$ is a family of independent Gaussian random variables with mean zero and covariance β. Indeed, if $\mathrm{d}\mu(Z^k)$ denotes the measure on $\mathbb{R}^{\Lambda_{N-k-1}}$ with covariance β and

$$\mathrm{d}\mu(\varphi) = \prod_{k=0}^{N-1} \mathrm{d}\mu(Z^k) \tag{2.3}$$

we find that

$$\int \varphi(x)\varphi(y)\,\mathrm{d}\mu(\varphi) = \sum_{k=K(x,y)-1}^{N-1} \beta = \beta C(x, y). \tag{2.4}$$

The renormalisation group transformations for this model consists of successively integrating out $Z^0, Z^1, \ldots$ in $\mathrm{e}^{-V(\varphi)}\,\mathrm{d}\mu(\varphi)$. We define ρ_n on $\mathbb{R}^{\Lambda_{N-n}}$ inductively by

$$\rho_0(\varphi) = \mathrm{e}^{-V(\varphi)}$$
$$\rho_{n+1}(\varphi) = \int \rho_n(\varphi([L^{-1}\cdot]) + Z^n([L^{-1}\cdot]))\,\mathrm{d}\mu(Z^n)/[\varphi = 0] \tag{2.5}$$

where $[\varphi = 0]$ is the numerator with $\varphi = 0$. Then we find that the effective measure after n steps is

$$\left(\int \rho_0(\varphi)\,\mathrm{d}\mu(Z^0)\ldots\mathrm{d}\mu(Z^{n-1})\right)\mathrm{d}\mu(\varphi^n) = \text{constant } \rho_n(\varphi^n)\,\mathrm{d}\mu(\varphi^n) \tag{2.6}$$

where:

$$\varphi^n(x) = \sum_{k=n}^{N-1} Z^k([L^{n-k-1}x]) \qquad \mathrm{d}\mu(\varphi^n) = \prod_{k=n}^{N-1} \mathrm{d}\mu(Z^k). \tag{2.7}$$

We will not find it necessary or useful to introduce effective potentials V_n, so $\rho_n = \mathrm{e}^{-V_n}$.

Next we claim that ρ_n has a product structure of the form:

$$\rho_n(\varphi) = \prod_{x\in\Lambda_{N-n}} T_n(\varphi(x)). \tag{2.8}$$

Here T_n is a function on $\mathbb{R}$ defined by

$$T_0(\varphi) = \mathrm{e}^{-2z\cos\varphi}$$
$$T_{n+1}(\varphi) = \int [T_n(\varphi + Z)]^m\,\mathrm{d}\mu(Z)/[\varphi = 0] \tag{2.9}$$

where $m = L^2$ and $\mathrm{d}\mu$ is Gaussian with covariance β. This is immediate for $n = 0$. Assuming it is true for n, we have

$$\rho_n(\varphi([L^{-1}\cdot]) + Z^n([L^{-1}\cdot])) = \prod_{u\in\Lambda_{N-n-1}} [T_n(\varphi(u) + Z^n(u))]^m \tag{2.10}$$

and substituting this into (2.5) establishes (2.8) for $n+1$.

Since T_n is periodic with period 2π, it has a Fourier series:

$$T_n(\varphi) = \sum_{p\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}p\varphi} t_n(p). \tag{2.11}$$

Theorem 1. Let β be sufficiently large and $z \leqslant \mathrm{e}^{-\beta}$ and $\beta' = \beta/6$, then for $n = 0, 1, 2, \ldots$

$$|t_n(p)| \leqslant \mathrm{e}^{-(n+1)\beta'|p|} \qquad p \neq 0$$
$$|t_n(0) - 1| \leqslant \mathrm{e}^{-(n+1)\beta'}. \tag{2.12}$$

Remark. As a consequence, $T_n(\varphi)$ converges to 1 as $n \to \infty$ exponentially fast and uniformly in φ. In this sense, $\rho_n(\varphi)$ converges to 1. This is the infrared asymptotic freedom.

Proof. We first establish the result for $n=0$. For $|\mathrm{Im}\ \varphi| \leqslant \beta$ we have $|2z\cos\varphi| \leqslant 2z\,e^{\beta} \leqslant 2$, and so $|T_0(\varphi)| \leqslant e^2$. Then in

$$t_0(p) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ip\varphi} T_0(\varphi)\,d\varphi \tag{2.13}$$

we may use the periodicity to deform the contour by $\varphi \to \varphi - i\beta$ for $p>0$ and $\varphi \to \varphi + i\beta$ for $p<0$. This gives the bound $|t_0(p)| \leqslant e^{-\beta|p|+2} \leqslant e^{-\beta'|p|}$ for $p \neq 0$. For $p=0$ we keep φ real. Then $|2z\cos\varphi| \leqslant e^{-\beta}$, hence $|T_0(\varphi)-1| \leqslant e^{-\beta+1} \leqslant e^{-\beta'}$, and hence $|t_0(0)-1| \leqslant e^{-\beta'}$.

Now we show n implies $n+1$. First consider the un-normalised quantity

$$T^*_{n+1}(\varphi) = \int [T_n(\varphi+Z)]^m\,d\mu(Z). \tag{2.14}$$

So $T_{n+1}(\varphi) = T^*_{n+1}(\varphi)/T^*_{n+1}(0)$. Using $\int e^{ipZ}\,d\mu(Z) = e^{-\beta p^2/2}$, we find for the Fourier coefficients t^*_{n+1} of T^*_{n+1}:

$$t^*_{n+1}(p) = \left(\sum_{p_1+\ldots+p_m=p}\ \prod_{i=1}^{m} t_n(p_i)\right) e^{-\beta p^2/2}. \tag{2.15}$$

To estimate (2.15), we use the general bound for $a \geqslant 1$

$$\left|\sum_{p_1+\ldots+p_m=p}\ \prod_{i=1}^{m} e^{-a|p_i|}\right| \leqslant O(1)(1+|p|^m)\,e^{-a|p|}. \tag{2.16}$$

To see this, note that we always have $|p| \leqslant \sum_i |p_i|$. We divide the sum into a region with $|p| \leqslant \sum|p_i| \leqslant 2|p|$ and a region with $2|p| \leqslant \sum_i |p_i|$. The first region has volume less than $O(1)|p|^m$, and hence the bound holds. For the second region we may extract a factor $e^{-a|p|}$ before estimating $\sum\prod_i \exp(-a|p_i|/2) \leqslant O(1)$.

We use this bound in (2.15) with $|t_n(p)| \leqslant 2e^{-(n+1)\beta'|p|}$. We also use $p^2 \geqslant |p|$, and hence $\exp(-\beta p^2/2) \leqslant \exp(-3\beta'|p|)$. A factor $\exp(-\beta'|p|)$ dominates $(1+|p|^m)$, and so

$$|t^*_{n+1}(p)| \leqslant O(1)\,e^{-(n+3)\beta'|p|} \tag{2.17}$$

which we will use for $p \neq 0$.

For $p=0$, we use

$$t^*_{n+1}(0) - (t_n(0))^m = \sum_{p_1+\ldots+p_m=0}{}' \prod_i t_n(p_i) \tag{2.18}$$

where the primes indicates that at least one $p_i \neq 0$. For this p_i we may extract a factor $e^{-(n+1)\beta'/2}$ from our bound on $t_n(p_i)$, and proceed as before to get $|t^*_{n+1}(0) - t_n(0)^m| \leqslant O(1)\,e^{-(n+1)\beta'/2}$. Since also $|t_n(0)^m - 1| \leqslant O(1)\,e^{-(n+1)\beta'}$, we conclude that

$$|t^*_{n+1}(0) - 1| \leqslant O(1)\,e^{-(n+1)\beta'/2}. \tag{2.19}$$

Now we estimate for $p \neq 0$

$$t_{n+1}(p) = t^*_{n+1}(p)\left(\sum_p t^*_{n+1}(p)\right)^{-1}. \tag{2.20}$$

By (2.17) and (2.19), the denominator is $1+O(e^{-(n+1)\beta'/2}) \geqslant \frac{1}{2}$, and thus, by (2.17), $|t_{n+1}(p)| \leqslant O(1)\,e^{-(n+3)\beta'|p|} \leqslant e^{-(n+2)\beta'|p|}$, as required.

For $p=0$ we have

$$t_{n+1}(0) - 1 = -\sum_{p\neq 0} t^*_{n+1}(p)\left(\sum_p t^*_{n+1}(p)\right)^{-1}. \tag{2.21}$$

By (2.17) we get $|t_{n+1}(0)-1| \leqslant O(1)\,e^{-(n+3)\beta'} \leqslant e^{-(n+2)\beta'}$, as required.

3. Correlation functions

For the correlation functions we consider the expected values of $e^{i(\varphi, f)}$ with $\sum f = 0$. We confine ourselves to the simplest choice $f = \gamma(\delta_x - \delta_y)$, where δ_x is a δ function at $x \in \Lambda$. Thus we study

$$\mathscr{G}(x, y) = \int \exp(i\gamma(\varphi_x - \varphi_y))\, e^{-V(\varphi)}\, d\mu(\varphi)/[\gamma = 0]. \tag{3.1}$$

In the original model this has the interpretation of measuring the correlation between a charge $+\gamma$ at x and a charge $-\gamma$ at y. We will assume $0 \leq \gamma < \frac{1}{2}$, so these would be fractional charges (cf [1]).

With no potential ($V = 0$) we have

$$\mathscr{G}_0(x, y) = \exp(-\beta\gamma^2 K(x, y)) \tag{3.2}$$

independent of the volume. Thus $\mathscr{G}_0(x, y)$ is approximately $|x - y|^{-a}$ with $a = \beta\gamma^2/\log L$. Our goal is to show that $\mathscr{G}(x, y)$ has the same behaviour to leading order as $|x - y| \to \infty$.

We analyse $\mathscr{G}(x, y)$ by successively integrating out the short-distance modes. This takes one form for the first $K = K(x, y)$ steps, and another form thereafter.

We claim that for $n = 0, 1, \ldots, K$

$$\mathscr{G}(x, y) = \int F_n^+(\varphi^n([L^{-n}x]))F_n^-(\varphi^n([L^{-n}y]))\rho_n(\varphi^n)\, d\mu(\varphi^n)/[F = 1] \tag{3.3}$$

where $F_n^\pm$ are functions on $\mathbb{R}$ defined inductively by

$$\begin{aligned} F_0^\pm(\varphi) &= e^{\pm i\gamma\varphi} \\ F_{n+1}^\pm(\varphi) &= \int F_n^\pm(\varphi + Z)(T_n(\varphi + Z))^m\, d\mu(Z)/T_{n+1}^*(\varphi). \end{aligned} \tag{3.4}$$

This is the definition for $n = 0$. Assuming it is true for n, we insert $\varphi^n = \varphi^{n+1}([L^{-1}\cdot]) + Z^n(L^{-1}\cdot])$ and $d\mu(\varphi^n) = d\mu(\varphi^{n+1})\, d\mu(Z^n)$ and (2.10) into (3.3). Now $d\mu(Z^n) = \Pi_u\, d\mu(Z^n(u))$ and in the integral over $Z^n(u)$ we may identify $T_{n+1}^*(\varphi^{n+1}(u))$ except when $u = [L^{-(n+1)}x]$ or $u = [L^{-(n+1)}y]$, when we get respectively $F_{n+1}^\pm(\varphi^{n+1}(u))$ $T_{n+1}^*(\varphi^{n+1}(u))$. Dividing by $\Pi_u T_{n+1}^*(0)$ changes the T_{n+1}^* to T_{n+1}, and identifying $\Pi_u T_{n+1}(\varphi^{n+1}(u)) = \rho_{n+1}(\varphi^{n+1})$ gives the result for $n + 1$.

In the following it will be convenient to define $R_n^\pm$ by

$$F_n^\pm(\varphi) = e^{\pm i\gamma\varphi} R_n^\pm(\varphi) \tag{3.5}$$

and then

$$\begin{aligned} R_0^\pm(\varphi) &= 1 \\ R_{n+1}^\pm(\varphi) &= \int e^{\pm i\gamma Z} R_n^\pm(\varphi + Z)(T_n(\varphi + Z))^m\, d\mu(Z)/T_{n+1}^*(\varphi). \end{aligned} \tag{3.6}$$

Next we claim that for $n = K, K + 1, \ldots, N$ so $[L^{-n}x] = [L^{-n}y]$ we have

$$\mathscr{G}(x, y) = \int G_n(\varphi^n([L^{-n}x]))\rho_n(\varphi^n)\, d\mu(\varphi^n)/[G = 1]. \tag{3.7}$$

Here G_n is defined by

$$G_K(\varphi)=F_K^+(\varphi)F_K^-(\varphi)=R_K^+(\varphi)R_K^-(\varphi)$$
$$G_{n+1}(\varphi)=\int G_n(\varphi+Z)(T_n(\varphi+Z))^m\,\mathrm{d}\mu(Z)/T_{n+1}^*(\varphi). \tag{3.8}$$

This is true for $n=K$ by (3.3) and the proof for all n follows as before. Note that for $n=N$ there are no variables left and we have just $\mathscr{G}(x,y)=G_N(0)$.

Note also that $R_n^\pm$ and G_n are periodic with period 2π ($F_n^\pm$ was not). Thus we may study their behaviour through their Fourier coefficients denoted $r_n^\pm(p)$, $g_n(p)$. One can show that $r_n^\pm(p)$ is real, and that $r_n^+(p)=r_n^-(-p)$.

The estimates to follow will involve the quantity

$$\delta=\exp(-\beta\gamma^2/2). \tag{3.9}$$

We consider γ fixed with $0\leqslant\gamma<\frac{1}{2}$, and let

$$\alpha=(1-2\gamma)\beta/18 \tag{3.10}$$

we always assume that β (and hence α) is sufficiently large.

Lemma 1. For $n=0,1,\ldots,K$:

(a) define c_n by $r_n^\pm(0)=\delta^n(1+c_n)$; then

$$|c_n-c_{n-1}|\leqslant \mathrm{e}^{-(n+1)\alpha};$$

(b) for $p\neq 0$

$$|r_n^\pm(p)|\leqslant\delta^n\,\mathrm{e}^{-(n+1)\alpha|p|}.$$

Proof. Since $r_0(p)=\delta_{p,0}$, we have $c_0=0$ and the bounds are true for $n=0$ (by convention $c_{-1}=0$). We assume they are true up to n and prove them for $n+1$.

Define

$$S(\varphi)=R_{n+1}^\pm(\varphi)-\delta r_n^\pm(0) \tag{3.11}$$

Then the Fourier coefficients satisfy $s(p)=r_{n+1}^\pm(p)$ for $p\neq0$ and $s(0)=\delta^{n+1}(c_{n+1}-c_n)$. Thus we must show that

$$|s(p)|\leqslant\delta^{n+1}\,\mathrm{e}^{-(n+2)\alpha|p|}\qquad p\neq 0$$
$$|s(0)|\leqslant\delta^{n+1}\,\mathrm{e}^{-(n+2)\alpha}. \tag{3.12}$$

Next we remove the denominators defining $S^*(\varphi)=S(\varphi)T_{n+1}^*(\varphi)$, so that

$$S^*(\varphi)=\int \mathrm{e}^{\pm\mathrm{i}\gamma Z}R_n^\pm(\varphi+Z)(T_n(\varphi+Z))^m\,\mathrm{d}\mu(Z)-\delta r_n^\pm(0)\,T_{n+1}^*(\varphi). \tag{3.13}$$

This has Fourier coefficients

$$s^*(p)=\left(\sum_{\substack{q,p_i\\ q+\sum p_i=p}} r_n^\pm(q)\prod_i t_n(p_i)\right)\exp(-\beta(p\pm\gamma)^2/2)-\delta r_n^\pm(0)t_{n+1}^*(p). \tag{3.14}$$

We claim that

$$|s^*(p)|\leqslant \mathrm{O}(1)\delta^{n+1}\,\mathrm{e}^{-(n+3)\alpha|p|}. \tag{3.15}$$

Indeed by $|r_n^\pm(q)\le 2\delta^n\,\mathrm{e}^{-(n+1)\alpha|q|}$ (since $|c_n|\le 1$) and $|t_n(p)|\le 2\,\mathrm{e}^{-3\alpha(n+1)|p|}$ (since $\beta'\ge 3\alpha$) and (2.16), the term in parentheses is $\mathrm{O}(1)\delta^n\mathrm{e}^{-n\alpha|p|}$. For the exponent we use

$$\tfrac{1}{2}\beta(p\pm\gamma)^2\ge\tfrac{1}{2}\beta(|p|(1-2\gamma)+\gamma^2)\ge 3\alpha|p|+\beta\gamma^2/2. \tag{3.16}$$

These combine to give the claimed bound for the first term in $\sigma^*(p)$. For the second term we use (2.17) to get the same bound.

For $p=0$ there is a cancellation, and we have

$$\sigma^*(0)=\Bigg(\sum_{\substack{q\neq 0,p_i\\ q+\sum p_i=0}} r_n^*(q)\prod_i t_n(p_i)\Bigg)\exp(-\beta\gamma^2/2). \tag{3.17}$$

We repeat the bound, but use the fact that at least one $p_i\neq 0$ to extract a factor $\mathrm{e}^{-2\alpha(n+1)}$. Thus we have

$$|\sigma^*(0)|\le \mathrm{O}(1)\delta^{n+1}\,\mathrm{e}^{-2\alpha(n+1)}. \tag{3.18}$$

Now consider $S(\varphi)=S^*(\varphi)T_{n+1}^*(\varphi)$ in the region $|\mathrm{Im}\,\varphi|\le(n+2)\alpha$. By (3.15) and (3.18) we have that $S^*(\varphi)$ is analytic in the region and bounded by $\mathrm{O}(1)\delta^{n+1}\,\mathrm{e}^{-\alpha}$. By (2.17) and (2.19), $1-T_{n+1}^*(\varphi)$ is also analytic and bounded by $\frac{1}{2}$. Thus $S(\varphi)$ is analytic and bounded by $\mathrm{O}(1)\delta^{n+1}\,\mathrm{e}^{-\alpha}$. By a contour deformation argument we conclude that for $p\neq 0$, $\sigma(p)\le(\mathrm{O}(1)\delta^{n+1}\,\mathrm{e}^{-\alpha})(\mathrm{e}^{-(n+2)\alpha|p|})$ and this implies (3.12) for $p\neq 0$.

If we keep φ real we can bound $S^*(\varphi)$ by $\mathrm{O}(1)\delta^{n+1}\,\mathrm{e}^{-(n+3)\alpha}$. Hence $S(\varphi)$ has a bound of the same form, and so does $\sigma(0)$ and this gives (3.12) for $p=0$.

Lemma 2. For $n=K, K+1,\dots,N$ and $p\neq 0$

$$|g_n(p)|\le\delta^{2K}\,\mathrm{e}^{-(n+1)\alpha|p|/2}. \tag{3.19}$$

Proof. First for $n=K$ we have

$$g_K(p)=\sum_{q+q'=p} r_K^+(q)r_K^-(q'). \tag{3.20}$$

Then $|r_K^\pm(q)|\le 2\delta^K\,\mathrm{e}^{-(K+1)\alpha|q|}$ and (2.16) give the bound.

Now we show n implies $n+1$. We define

$$S(\varphi)=G_{n+1}(\varphi)-g_n(0) \tag{3.21}$$

the Fourier coefficients are $\sigma(p)=g_{n+1}(p)$ for $p\neq 0$, so it suffices to prove

$$|\sigma(p)|\le\delta^{2K}\,\mathrm{e}^{-(n+2)\alpha|p|/2}. \tag{3.22}$$

Next let $S^*(\varphi)=S(\varphi)T_{n+1}^*(\varphi)$ so that

$$S^*(\varphi)=\int G_n(\varphi+Z)(T_n(\varphi+Z))^m\,\mathrm{d}\mu(Z)-g_n(0)T_{n+1}^*(\varphi). \tag{3.23}$$

This function has Fourier coefficients:

$$\sigma^*(p)=\Bigg(\sum_{\substack{q\neq 0,p_i\\ q+\sum p_i=0}} g_n(q)\prod_i t_n(p_i)\Bigg)\exp(-\beta p^2/2). \tag{3.24}$$

(The $q=0$ term was canceled by $g_n(0)t_{n+1}^*(p)$.) Following the analysis of (3.15)-(3.18) with minor modifications, we get $|\sigma^*(p)|\le\mathrm{O}(1)\delta^{2K}\,\mathrm{e}^{-(n+3)\alpha|p|/2}$ and $|\sigma^*(0)|\le\mathrm{O}(1)\delta^{2K}\,\mathrm{e}^{-\alpha(n+1)}$. Then $S^*(\varphi)$ is analytic in $|\mathrm{Im}\,\varphi|\le(n+2)\alpha/2$ and bounded by $\mathrm{O}(1)\delta^{2K}\,\mathrm{e}^{-\alpha/2}$, hence $S(\varphi)=S^*(\varphi)/T_{n+1}^*(\varphi)$ is also. Therefore $|\sigma(p)|\le(\mathrm{O}(1)\delta^{2K}\,\mathrm{e}^{-\alpha/2})(\mathrm{e}^{-(n+2)\alpha|p|/2})$; so (3.22) follows.

Now can control the infinite-volume limit and asymptotic behaviour for $\mathscr{G}(x, y)$.

Theorem 2.

(a) $\mathscr{G}(x, y) = \lim_{N\to\infty} \mathscr{G}(x, y)$ and $c_\infty = \lim_{n\to\infty} c_n$ exist.

(b) For this limit

$$\lim_{|x-y|\to\infty} \mathscr{G}(x, y)/\mathscr{G}_0(x, y) = (1+c_\infty)^2 \neq 0. \tag{3.25}$$

Proof.

(a) We have with, $K = K(x, y)$,

$$\mathscr{G}(x, y) = G_N(0) = \sum_{n=K}^{N-1} (G_{n+1}(0) - G_n(0)) + G_K(0) \tag{3.26}$$

and the only N dependence is in the upper limit of the sum. By lemma 2 we have

$$|G_n(Z) - G_n(0)| = \left| \sum_{p\neq 0} (e^{ipZ} - 1) g_n(p) \right| \leqslant O(1)\delta^{2K} e^{-(n+1)\alpha/2}. \tag{3.27}$$

This in turn gives

$$\begin{aligned} |G_{n+1}(0) - G_n(0)| &= \left| \int (G_n(Z) - G_n(0)) T_n(Z)^m \, d\mu(Z)/T^*_{n+1}(0) \right| \\ &\leqslant O(1)\delta^{2K} e^{-(n+1)\alpha/2}. \end{aligned} \tag{3.28}$$

Hence the $N\to\infty$ limit in (3.26) exists. The limit for c_n follows from $c_n = \sum_{k=1}^n (c_k - c_{k-1})$ and lemma 1(a). Note also that $|c_\infty| \leqslant e^{-\alpha}$, and so is small.

(b) Since $\mathscr{G}_0(x, y) = e^{-\beta\gamma^2 K} = \delta^{2K}$, we have

$$\mathscr{G}(x, y)/\mathscr{G}_0(x, y) = \sum_{n=K}^{\infty} \delta^{-2K}(G_{n+1}(0) - G_n(0)) + \delta^{-2K} G_K(0).$$

As $|x-y|\to\infty$, we have $K\to\infty$, and the sum converges to zero by (3.28). For the remaining term we use lemma 1(b) to get

$$\lim_{K\to\infty} \delta^{-K} R^\pm_K(0) = \lim_{K\to\infty} \left((1+c_K) + \sum_{p\neq 0} \delta^{-K} r^\pm_K(p) \right) = 1 + c_\infty$$

and hence, since $G_K(0) = R^+_K(0) R^-_K(0)$, we obtain

$$\lim_{K\to\infty} \delta^{-2K} G_K(0) = (1+c_\infty)^2.$$

Note added. After this paper was completed I discovered the paper by Benefatto *et al* [13] in which very similar results were obtained by Mayer expansion techniques. The present method is quite different and somewhat simpler.

References

[1] Fröhlich J and Spencer T 1980 Phase diagrams and critical properties of (classical) Coulomb systems *Proc. 1980 Erice Summer School* ed A Wightman and G Velo

[2] Gallavotti G and Nicolo F 1985 The 'screening phase transitions' in the two-dimensional Coulomb gas *J. Stat. Phys.* **39** 133
[3] Brydges D 1978 A rigorous approach to Debye screening in dilute Coulomb systems *Commun. Math. Phys.* **58** 313
[4] Brydges D and Federbush P 1980 Debye screening *Commun. Math. Phys.* **73** 197
[5] Kosterlitz J and Thouless D 1973 Ordering, metastability, and phase transitions in two-dimensional systems *J. Phys. C: Solid State Phys.* **6** 1181
[6] Fröhlich J and Spencer T 1981 The Kosterlitz-Thouless transition in two-dimensional Abelian spin systems and the Coulomb gas *Commun. Math. Phys.* **81** 527
[7] Wiegmann P 1978 One-dimensional Fermi system and plane XY model *J. Phys. C: Solid State Phys.* **11**
[8] Gawedzki K and Kupianinen A 1983 Block spin renormalisation group for dipole gas and $(\nabla\varphi)^4$ *Ann. Phys.* **147** 198
[9] Gawedzki K and Kupiainen A 1983 Rigorous renormalisation group and asymptotic freedom *Scaling and Self-similarity in Physics* ed J Fröhlich (Basel: Birkhauser)
[10] Gawedzki K and Kupiainen A 1986 Asymptotic freedom beyond perturbation theory *Critical Phenomena, Random Systems, Gauge Theories* ed K Osterwalder and R Stora (Amsterdam: North-Holland)
[11] Gawedzki K and Kupiainen A 1985 Nontrivial continuum limit of a $(\varphi^4)_4$ model with negative coupling constant *Nucl. Phys.* B **257** 474
[12] Gawedzki K and Kupiainen A 1986 Continuum limit of the hierarchical O(N) non-linear σ-model *Commun. Math. Phys.* **106** 533
[13] Benefatto G, Gallavotti G and Nicolo F 1986 *Commun. Math. Phys.* **106** 277